

Simulation of multi-field coupled multiple crack interactions based on the fracture phase-field method

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Abstract. Crack propagation and multi-crack interaction under multi-field coupling conditions constitute a central problem in material failure research, and conventional fracture mechanics approaches are generally inadequate for capturing such complex behavior. This work develops a phase-field fracture framework for simulating multi-field coupled multi-crack interaction, with particular emphasis on crack growth driven by coupled temperature/concentration and displacement fields. In the theoretical formulation, cracks are represented in a diffuse manner through the phase-field variable, and the degradation of thermal conductivity and diffusivity due to fracture damage is explicitly incorporated. Governing equations are established for thermo-mechanical coupling and for electrochemical concentration-displacement coupling, and the coupled system is solved numerically using an implicit finite element scheme. The model is first validated against mode I crack propagation and three-point bending fracture problems, where the numerical results agree closely with analytical solutions and previously reported data. The framework is then extended to multi-crack interaction scenarios, including parallel crack interaction and discharge processes in lithium electrodes. The results indicate that: (1) as the crack spacing decreases, the initial propagation angle increases and the interaction effect between cracks becomes more pronounced; (2) during glass quenching, thermally induced crack propagation exhibits a hierarchical pattern consistent with experimental observations; (3) during discharge in lithium electrodes, concentration gradients drive crack nucleation and growth, with crack propagation near the notch affected by a shielding effect. The proposed model is capable of effectively capturing multi-crack interaction and propagation behavior under multi-field coupling conditions, thereby offering a theoretical basis and a computational tool for multi-field fracture analysis of engineering structures.

Keywords: crack propagation, crack arrest, phase-field modeling, multi-field coupling

1. Introduction

Crack propagation is a key mechanism of material failure. In practical engineering applications, materials are often subjected to multiple physical fields simultaneously—such as thermal shock in structural components under high-temperature environments, electrochemical-mechanical coupling during the charging and discharging of lithium-ion batteries, and irradiation damage in nuclear reactor components. Under such multi-field coupled conditions, crack propagation behavior often exhibits complex nonlinear characteristics, posing

significant challenges to the safety and reliability of engineering structures. These multi-field coupling conditions significantly influence crack initiation, propagation paths, and interactions among multiple cracks, leading to more complex failure modes that are difficult to accurately predict using traditional fracture mechanics methods.

In recent years, the fracture phase-field method, as an advanced numerical simulation technique, has attracted widespread attention due to its ability to more efficiently handle complex behaviors such as crack initiation, propagation, and branching [1, 2]. Compared with traditional extended finite element methods and boundary element methods, the phase-field method characterizes cracks by introducing continuous phase-field variables, thereby eliminating the need for special treatment at the crack tip and enabling the natural capture of dynamic crack propagation and bifurcation phenomena [3, 4]. A comprehensive theoretical framework has been established for applying the phase-field method to fracture problems; this method drives the evolution of the phase-field variable by minimizing the total energy of the system, thereby enabling natural crack propagation [5]. The phase-field method has also made significant progress in simulating multiphysics coupling and multi-crack interactions, becoming a key numerical tool for studying complex fracture problems [6-8]. Because this method can naturally describe complex behaviors such as crack propagation, branching, and coalescence, it has been widely applied to multiphysics coupling problems—including thermo-mechanical, electro-chemical-mechanical, and fluid-solid interactions—as well as problems involving the evolution of multiple cracks [5, 9, 10]. Most studies have focused on modeling coupling mechanisms and crack behavior in specific fields [4, 11-13], such as thermal cracks under thermo-mechanical coupling conditions, corrosion cracks under electro-chemical-mechanical coupling conditions, or hydraulic fracturing under fluid-structure coupling conditions [14-16].

Despite significant progress in fracture simulation using the phase-field method, its application under multi-field coupling conditions still faces numerous challenges. The existing literature has notable shortcomings in the following areas: (1) There has been little systematic consideration of the feedback mechanisms by which material damage affects physical field parameters—such as thermal conductivity and diffusion coefficients—under strong multi-field coupling [17]; (2) Quantitative descriptions of interactions among multiple cracks, as well as their numerical and experimental validation in complex engineering scenarios, remain relatively limited [18, 19]; (3) The application of efficient solution strategies, such as implicit finite elements, in the efficient simulation of strongly coupled multi-field systems and multiple cracks needs to be expanded.

2. Computational methods

2.1. Phase-field description of fracture

Building upon our prior work [20], computational methods are developed. As shown in Figure 1(a), we consider an isotropic body Ω with an outer boundary $\partial\Omega$, where the boundary can be expressed as the sum of two types of boundary conditions for each physical field: $\partial\Omega = \partial\Omega_t \cup \partial\Omega_u$, $\partial\Omega = \partial\Omega_{J_T} \cup \partial\Omega_T$, $\partial\Omega = \partial\Omega_{J_c} \cup \partial\Omega_T$. Displacement u , temperature T , and concentration c are the fundamental variables describing the deformation of the body, heat conduction, and concentration diffusion. J_T is the heat flux per unit area per unit time, J_c is the ion flux per unit area per unit time, and t is the tensile force acting on $\partial\Omega_t$. The gray region within the deformable body Ω represents the internal heat source γ_T or ion source γ_c . The discontinuity Γ inside Figure 1(a) represents a non-continuous crack within the body. For phase-field method calculations of fracture problems, the non-continuous crack can be approximated by a dispersed phase-field

crack through the spatial continuous variation of the phase-field variable d , as shown in Figure 1(b). The scalar d represents the intact state of the material when $d = 0$, and the fractured state when $d = 1$.

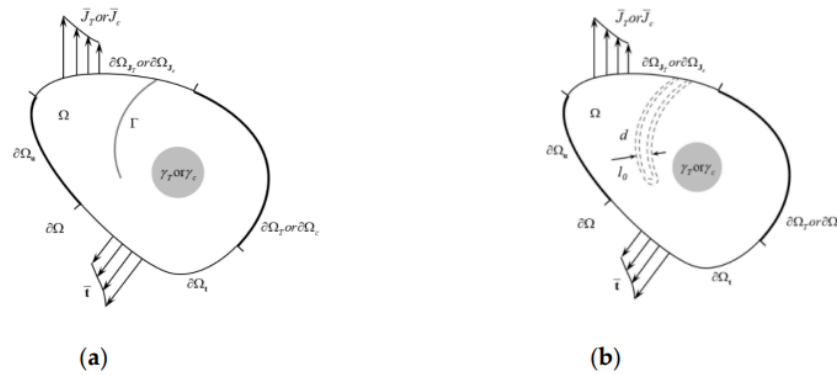


Figure 1. Schematic illustration of a fracture problem in a deformable body under the coupling of temperature/concentration fields and displacement fields. (a) An internal fracture produces a discontinuity Γ . (b) The internal crack is described as a dispersed feature via the phase-field variable d , and its dispersion width is controlled by the scale parameter l_0

2.2. Governing equations and constitutive relations

2.2.1. Temperature/concentration field

The temperature and concentration fields are governed by the classical heat conduction equation and diffusion equation, respectively. These two fields share governing equations that are formally similar. The heat conduction equation for the temperature field is given by:

$$\rho c_T \dot{T} + \nabla \cdot J_T = \gamma_T \quad (1)$$

The diffusion equation is given by:

$$\dot{c} + \nabla \cdot J_c = \gamma_c \quad (2)$$

For the temperature field, Fourier's law is adopted, in which the heat flux is proportional to the temperature gradient:

$$J_T = -k \nabla T \quad (3)$$

where k is the thermal conductivity.

For the concentration field, Fick's law is adopted, in which the flux is proportional to the concentration gradient. Thus:

$$J_c = -D \nabla c \quad (4)$$

where D is the diffusion coefficient.

If the influence of local fracture damage on thermal conductivity and diffusivity is taken into account, then:

$$k = (1 - d)^2 k_0, \quad D = (1 - d)^2 D_0 \quad (5)$$

where k_0 and D_0 are the intrinsic thermal conductivity and diffusivity of the material.

2.2.2. Displacement field

For the displacement field, the motion of a material point can be described by the momentum equation. Under quasi-static conditions, the governing equation is:

$$\nabla \cdot \sigma = 0 \quad (6)$$

Only small deformations are considered here, and the small strain tensor is expressed as:

$$\varepsilon = \frac{1}{2} (\nabla u + (\nabla u)^T) \quad (7)$$

For thermo-mechanical coupling problems in brittle materials, changes in temperature and concentration induce structural deformation and thereby generate stress. In this case, the strain can be decomposed in an additive form as:

$$\varepsilon = \varepsilon_e + \varepsilon_T \quad (8)$$

where ε_e and ε_T are the elastic strain tensor and thermal strain tensor, respectively. For the chemical-displacement coupling problem in brittle materials, the strain can be decomposed in an additive form as:

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad (9)$$

where ε_c is the strain induced by the concentration field.

Assuming that ε_T and ε_c are proportional to temperature and concentration, respectively, we have:

$$\varepsilon_T = \alpha T I, \quad \varepsilon_c = \theta c I \quad (10)$$

where α and θ are the linear thermal expansion coefficient and the linear concentration expansion coefficient of the material, respectively, and I is the identity tensor.

According to the linear elastic constitutive relation, the stress tensor σ is given by:

$$\sigma = C : \varepsilon_e, \quad C = (1 - d)^2 C_0 \quad (11)$$

where C_0 is the intrinsic (undamaged) isotropic elastic stiffness tensor of the material, and C is the stiffness tensor degraded by the phase-field variable d . The degradation form is similar to that adopted for the thermal conductivity.

2.2.3. Phase field

The fracture phase field is driven by structural deformation, and its evolution follows the principle of energy minimization. Since fracture is directly induced by structural deformation and the energy directly released upon fracture is the structural deformation energy, the evolution of the phase field can be regarded as minimizing the sum of the deformation energy and the fracture energy of the system.

According to classical Griffith theory, the crack surface energy can be obtained by integrating the critical energy release rate G_c over the crack surface Γ . In the phase-field approach, the crack is represented by the phase-field variable d as a diffuse crack. The fracture energy Ψ_Γ can then be expressed in a volume integral form associated with d [8]:

$$\Psi_\Gamma = \int_\Omega G_c \left[\frac{d^2}{2l_0} + \frac{l_0}{2} |\nabla d|^2 \right] dV \quad (12)$$

The parameter l_0 controls the width of the diffuse crack. As l_0 approaches 1, the function more closely resembles a discontinuous function that can represent a sharp crack.

The elastic energy Ψ_e can be expressed as the volume integral of the elastic energy density ψ_e over the body Ω :

$$\Psi_e = \int_\Omega \psi_e dV \quad (13)$$

Taking into account the constitutive relation, the elastic energy density is expressed as:

$$\psi_e(d, u) = \frac{1}{2} \sigma : \varepsilon_e \quad (14)$$

Therefore, the evolution of the phase-field variable d follows:

$$\frac{\partial(\psi_e + \psi_d)}{\partial d} = 0 \quad (15)$$

Based on the constitutive relation and the phase-field boundary condition $\nabla d \cdot n = 0$, the phase-field evolution equation can be derived as:

$$\frac{G_c}{l_0} [d - l_0^2 \nabla^2 d] = 2(1 - d)\psi_{e0} \quad (16)$$

where

$$\psi_{e0} = \frac{1}{2} \varepsilon_e : C_0 : \varepsilon_e \quad (17)$$

is the elastic potential energy d before it is reduced by the variable d .

In the phase-field governing equation, the key term driving the evolution of d is the elastic energy density ψ_{e0} . To account for the asymmetry between tension and compression in driving the phase-field evolution, the elastic energy density is modified as:

$$\psi_{e0} = \begin{cases} \psi_{e0}, & tr(\varepsilon_e) \geq 0. \\ \psi_{e0} - \frac{1}{2} K_n tr(\varepsilon_e)^2, & tr(\varepsilon_e) < 0. \end{cases} \quad (18)$$

where $K_n = \lambda + \frac{2\mu}{n}$, $n = 2, 3$ is the bulk modulus for an n -dimensional problem, and λ and μ are the Lamé constants.

To prevent crack healing when ψ_{e0} decreases and to ensure that crack propagation is irreversible, the local maximum historical energy densit

$$H(u(x, s), \theta(x, s)) = \max_{s \in [0, t]} \psi_{e0}(u(x, s)) \quad (19)$$

is adopted to replace ψ_{e0} . In addition, to improve the stability of multi-field-coupled fracture, a viscous term is introduced, which is determined by the viscosity coefficient η and the rate of phase-field evolution. In this case, the evolution equation for the phase field is

$$\eta \dot{d} + \frac{G_c}{l_0} [d - l_0^2 \nabla^2 d] = 2(1 - d)H \quad (20)$$

The strong form of the governing equations for the above multi-field coupled fracture problem can be summarized as:

$$\begin{aligned} \rho c_T \dot{T} + \nabla \cdot J_T &= \gamma_T \quad \text{in } \partial\Omega, \\ \dot{c} + \nabla \cdot J_c &= \gamma_c \quad \text{in } \partial\Omega, \\ \nabla \cdot \sigma &= 0 \quad \text{in } \Omega, \\ \eta \dot{d} + \frac{G_c}{l_0} [d - l_0^2 \nabla^2 d] &= 2(1 - d)H \quad \text{in } \Omega, \\ T &= \bar{T} \quad \text{on } \partial\Omega_T, \\ J_T \cdot n &= \bar{J}_T \quad \text{on } \partial\Omega_{J_T}, \\ c &= \bar{c} \quad \text{on } \partial\Omega_c, \\ J_c \cdot n &= \bar{J}_c \quad \text{on } \partial\Omega_{J_c}, \\ u &= \bar{u} \quad \text{on } \partial\Omega_u, \end{aligned} \quad (21)$$

$$\begin{aligned}\sigma \cdot n &= t \quad \text{on } \partial\Omega_t, \\ \nabla d \cdot n &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

The top horizontal line indicates the value on the boundary.

2.3. Solving coupled equations

To numerically solve the fracture problems described above under thermal coupling or chemical-displacement field coupling, we employ an implicit finite element method to solve the equations. This section primarily introduces the numerical computational approach [20].

2.3.1. Finite element spatial discretization

The weak form of the above strong Galerkin form is:

$$\begin{aligned}\int_{\Omega} [\rho c_T \dot{T} \delta T] dV - \int_{\Omega} [J_T \cdot \nabla (\delta T)] dV - \int_{\Omega} [\gamma_T \delta T] dV + \int_{\partial\Omega_{J_T}} [\bar{J}_T \delta T] dA &= 0 \\ \int_{\Omega} [\dot{c} \delta c] dV - \int_{\Omega} [J_c \cdot \nabla (\delta c)] dV - \int_{\Omega} [\gamma_c \delta c] dV + \int_{\partial\Omega_{J_c}} [\bar{J}_c \delta c] dA &= 0 \\ \int_{\Omega} [\sigma : \delta \varepsilon] dV - \int_{\partial\Omega_T} [\bar{t} \cdot \delta u] dV &= 0 \\ \int_{\Omega} [(\frac{G_c}{l_0} + 2H)(\delta d) d] dV + \int_{\Omega} [G_c l_0 \nabla d \cdot \nabla (\delta d)] dV + \int_{\Omega} [(\eta \dot{d} - 2H) \delta d] dV &= 0\end{aligned}\tag{22}$$

where δT , δc , δu , and δd are the variational trial functions for v . The above weak form can be solved implicitly by the finite element method. For two-dimensional problems, a linear four-node quadrilateral element is used to discretize the body Ω in space. In this case, all volume integrals become area integrals, and all surface integrals become line integrals. Using Voigt notation, the basic field variables can be interpolated from the nodal field variable values θ_I , c_I , u_I , and d_I :

$$\theta = \sum_{i=1}^4 N_i^\theta \theta_i, \quad c = \sum_{i=1}^4 N_i^c c_i, \quad \mathbf{u} = \sum_{i=1}^4 \mathbf{N}_i^u \mathbf{u}_i, \quad d = \sum_{i=1}^4 N_i^d d_i\tag{23}$$

where $N_I^u = \begin{bmatrix} N_I & 0 \\ 0 & N_I \end{bmatrix}$, $N_I^\theta = N_I^c = N_I^d = N_I$, N_I denotes the shape function associated with node I .

Correspondingly, the gradients of the field variables can be expressed as:

$$\nabla T = \sum_{i=1}^4 \mathbf{B}_i^T T_i, \quad \nabla c = \sum_{i=1}^4 \mathbf{B}_i^c c_i, \quad \varepsilon = \sum_{i=1}^4 \mathbf{B}_i^u \mathbf{u}_i, \quad \nabla d = \sum_{i=1}^4 \mathbf{B}_i^d d_i\tag{24}$$

where $B_I^u = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \end{bmatrix}$, $B_I^T = B_I^c = B_I^d = \begin{bmatrix} N_{I,x} \\ N_{I,y} \end{bmatrix}$.

Substituting the discretized field variables and their gradients into the above Galerkin weak form, and considering the arbitrariness of the variational quantities, the contribution of node I within a single element to the implicit computational residual can be expressed as:

$$\begin{aligned}R_I^T &= \int_{\Omega} [\rho c_T \dot{T} N_I] dV - \int_{\Omega} [J_T B_I^T] dV - \int_{\Omega} [N_I \gamma_T] dV + \int_{\partial\Omega_{J_T}} [N_I \bar{J}_T] dA \\ R_I^c &= \int_{\Omega} [\dot{c} N_I] dV - \int_{\Omega} [J_c B_I^c] dV - \int_{\Omega} [N_I \gamma_c] dV + \int_{\partial\Omega_{J_c}} [N_I \bar{J}_c] dA \\ R_I^u &= \int_{\Omega} [\sigma B_I^u] dV - \int_{\partial\Omega_1} [N_I^u \bar{t}] dV \\ R_I^d &= \int_{\Omega} [(\frac{G_c}{l_0} + 2H) N_I d] dV + \int_{\Omega} [G_c l_0 \nabla d B_I^d] dV + \int_{\Omega} [(\eta \dot{d} - 2H) N_I] dV\end{aligned}\tag{25}$$

After finite element discretization, all tensors in the governing equations have been transformed into matrix form.

2.3.2. Time discretization

The time integration of the transient terms in the temperature field, concentration field, and phase field is approximated using the implicit backward Euler method. The discrete rates of change for temperature, concentration, and phase field is given respectively by:

$$\dot{T} = \frac{T-T^n}{\Delta t}, \quad \dot{c} = \frac{c-c^n}{\Delta t}, \quad \dot{d} = \frac{d-d^n}{\Delta t} \quad (26)$$

Assuming that the current time is time step $n + 1$, the field variables in the above equation without superscripts are all values at the current time step, while those with the superscript n represent values from Δt time steps ago.

2.3.3. Linearization and implicit computation

The Newton method is used to linearize the nonlinear governing equations, and the solution is solved at each time step to ensure that $R^T = 0, R^c = 0, R^u = 0, R^d = 0$. To improve computational convergence and reduce computational complexity, an interleaved time-integration scheme is employed. When calculating the residual value of a particular field at the current time step, the values of other field variables are typically taken from the previous time step, as expressed by:

$$\begin{aligned} R^T(T, d | u^n) &= 0 \\ R^c(c, d | u^n) &= 0 \\ R^u(u, T, c | d^n) &= 0 \\ R^d(d | u^n) &= 0 \end{aligned} \quad (27)$$

Based on the coupling relationships among the field variables described above, the iterative strategy for implicitly solving the above system of equations can be established as:

$$\begin{aligned} R^{T(k+1)} &= R^{T(k)} - K^{TT(k+1)} \Delta T^{(k+1)} - K^{Td(k+1)} \Delta d^{(k+1)} \\ R^{c(k+1)} &= R^{c(k)} - K^{cc(k+1)} \Delta c^{(k+1)} - K^{cd(k+1)} \Delta d^{(k+1)} \\ R^{u(k+1)} &= R^{u(k)} - K^{uu(k+1)} \Delta u^{(k+1)} - K^{uT(k+1)} \Delta T^{(k+1)} - K^{uc(k+1)} \Delta c^{(k+1)} \\ R^{d(k+1)} &= R^{d(k)} - K^{dd(k+1)} \Delta d^{(k+1)} \end{aligned} \quad (28)$$

where the superscript $k + 1$ denotes the current iteration step, k denotes the previous iteration step, and K represents the stiffness matrix. At the element level, the corresponding tangent stiffness matrix is given by:

$$\begin{aligned} K_{IJ}^{TT} &= \frac{\partial R_I^T}{\partial T_J} = \int_{\Omega} [\frac{\rho c T}{\Delta t} (N_I)^T N_J] dV + \int_{\Omega} [(1-d)^2 k_0 (B_I^T)^T B_J^T] dV \\ K_{IJ}^{Td} &= \frac{\partial R_I^T}{\partial d_J} = \int_{\Omega} [-2(1-d) k_0 (B_I^T)^T (\nabla T) N_J^d] dV \\ K_{IJ}^{uu} &= \frac{\partial R_I^u}{\partial u_J} = \int_{\Omega} [(1-d)^2 (B_I^u)^T C_0 B_J^u] dV \\ K_{IJ}^{uT} &= \frac{\partial R_I^u}{\partial T_J} = \int_{\Omega} [(B_I^u)^T C_0 \alpha N_J] dV \\ K_{IJ}^{dd} &= \frac{\partial R_I^d}{\partial d_J} = \int_{\Omega} [G_c l_0 (B_I^d)^T B_J^d] dV + \int_{\Omega} [(\frac{G_c}{l_0} + 2H + \frac{\eta}{\Delta t}) (N_I^d)^T N_J^d] dV \end{aligned} \quad (29)$$

Since the use of an implicit calculation method with staggered iteration inevitably introduces computational errors, it is necessary to appropriately control the computational time step to ensure computational accuracy. Inertial effects are not considered here, but the time effects of heat conduction and concentration diffusion are taken into account; therefore, the time step is set to no more than $L^2 \rho c_T / k_0$ or L^2 / D_0 , where L is the mesh feature size.

3. Validation examples

This chapter primarily verifies the correctness of the computational model presented in this paper through simple quasi-static fracture crack propagation numerical examples.

3.1. Type I crack propagation

The specimen geometry and loading conditions for this numerical example are shown in the Figure 2 below. The specimen is a square plate with a pre-existing crack whose length is half the side length. The bottom edge is constrained in the vertical direction, and a displacement boundary condition is applied to the top edge. The maximum displacement load is 0.008 mm, applied linearly. The material parameters are $E = 208$ GPa, $\nu = 0.30$, and $G_c = 2.7$ N/mm². To examine the influence of the diffuse crack width parameter l_0 and the viscosity parameter η on the computed results, l_0 is set to two different values, 0.0075 mm and 0.0375 mm, while η takes three values, 1×10^{-6} kN·s/mm², 2×10^{-6} kN·s/mm², and 3×10^{-6} kN·s/mm², for comparison. Matching the l_0 values, the characteristic mesh size is taken as 0.0025 mm.

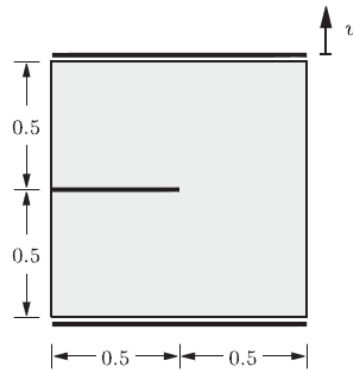


Figure 2. Geometry and boundary conditions for type I crack propagation

Figure 3 shows the final calculated crack propagation results when η is set to 2×10^{-6} kN·s/mm². The crack propagation paths calculated using the two methods are essentially consistent; when l_0 is smaller, the width of the diffuse crack is narrower. Figure 4 compares the force-displacement curves obtained using different parameters. A comparison of the curves reveals that when l_0 is smaller, the degree of material damage is lower before the fracture crack undergoes unstable propagation, and the force-displacement curve exhibits an approximately linear relationship; ultimately, l_0 significantly affects the specimen's ultimate load-bearing capacity. Regarding the use of different η values, when η is smaller, viscous dissipation during crack propagation is lower and the propagation rate is faster; when η approaches 0, the results approach ideal brittle fracture. Relatively speaking, when η is small, approximate brittle fracture behavior can be obtained. Additionally, Figure 4 compares the results of this study with the force-displacement curve calculation results for this problem by Miehe et al. 2010 [8] on this problem. The two sets of results show a high degree of agreement, confirming the validity of the computational model and program used in this study.

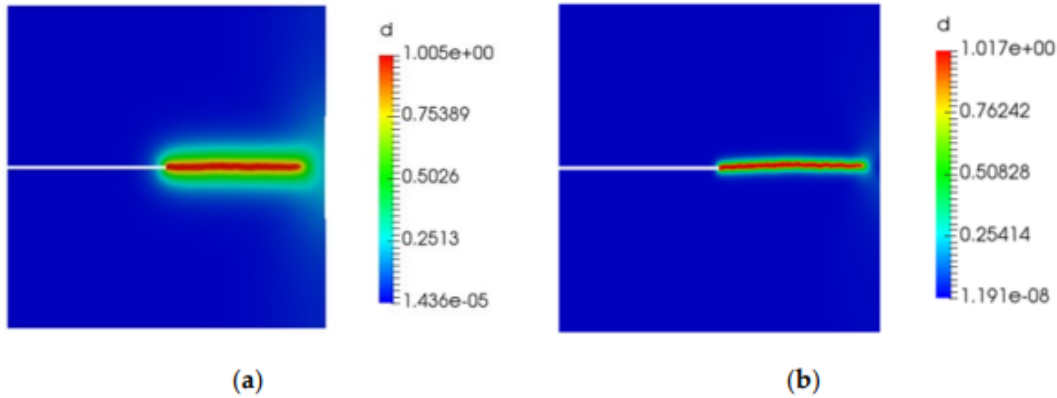


Figure 3. Contour plot of the final phase-field variable calculated using different values of l_0 : (a) $l_0 = 0.0375$ mm; (b) $l_0 = 0.0075$ mm

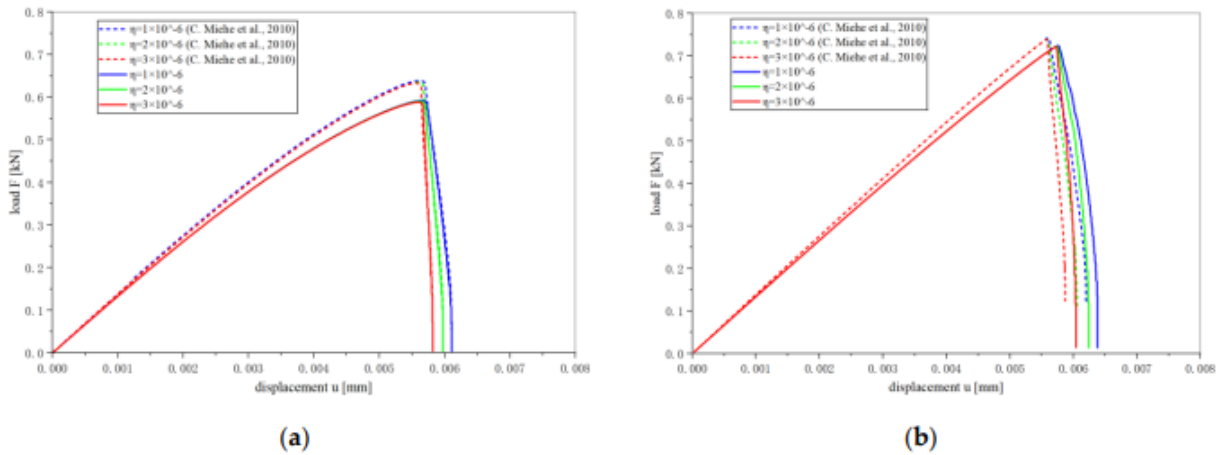


Figure 4. Force-displacement curves obtained using different η parameters and a comparison with the results from C. Miehe et al., 2010 : (a) $l_0 = 0.0375$ mm; (b) $l_0 = 0.0075$ mm

3.2. Three-point bending fracture of beams

This example simulates the three-point bending fracture of a beam. The geometry and loading conditions are shown in the Figure 5 below. The rectangular beam has a height of 2 mm and a length of 8 mm, with a triangular initial notch at the center of the bottom edge; the notch width is 0.2 mm. The left side of the beam bottom is constrained, the right side is simply supported, and loading is applied through a displacement boundary condition at the midpoint of the top surface. The maximum displacement load is 0.008 mm, applied linearly. The material parameters are $E = 20.8$ GPa, $\nu = 0.30$, $l_0 = 0.03$ mm. Near the crack path at the beam center, the mesh density is 0.01 mm, gradually transitioning to 0.04 mm toward both sides. Different values of η ($\eta = 2.5 \times 10^{-2}$ N·s/mm², 1.5×10^{-2} N·s/mm², 0.5×10^{-2} N·s/mm²) and G_c ($G_c = 0.6$ N/mm, 1.0 N/mm) are used to simulate this problem, so as to investigate the influence of these parameters on the failure process.

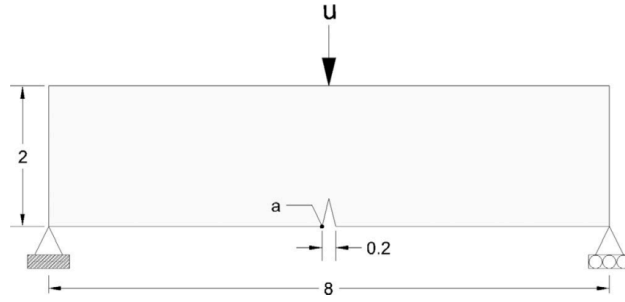


Figure 5. Geometry, boundary conditions, and loading conditions for the three-point bending example

The final beam failure mode obtained from the calculations is shown in Figure 6. Tensile stress concentrates at the location of the pre-cut notch, and a tensile crack continues to propagate along the notch. Figure 7 shows the force-displacement curves obtained using different parameters. As can be seen from the curves, beam fracture undergoes an initial unstable propagation stage followed by a subsequent stable propagation stage. After crack initiation, the crack propagates unsteadily due to the sudden and rapid release of localized energy; during this process, the force-displacement curve drops sharply. Subsequently, once the crack has propagated to a certain point, the propagation rate slows as the load continues to increase. The curves indicate that η has a more pronounced effect on the unstable propagation phase, while the force-displacement curves are essentially identical during the subsequent stable propagation phase. G_c influences the specimen's fracture limit, the endpoint of unstable crack propagation, and the stable crack propagation phase, but has little effect on the slope of the force-displacement curve during the unstable crack propagation phase.

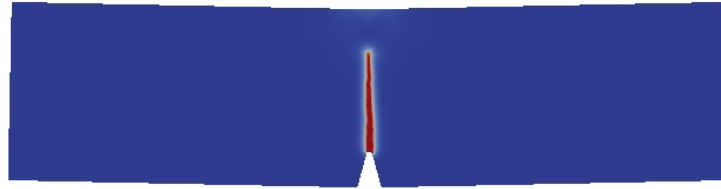


Figure 6. Contour plot of the final phase-field variables for the three-point bending fracture case

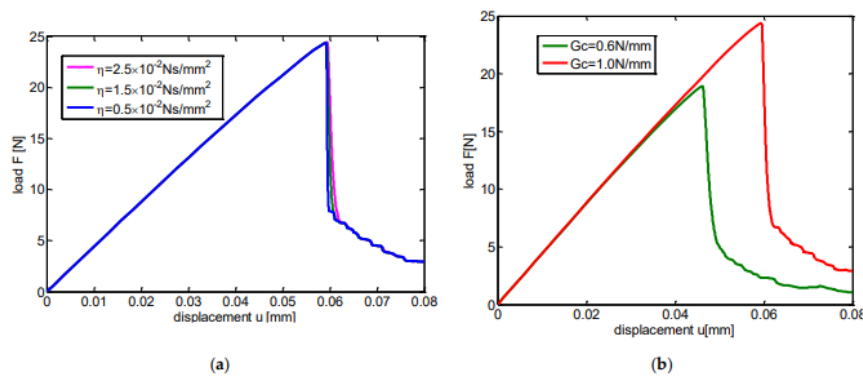


Figure 7. Force-displacement curves for the three-point bending fracture process calculated for different values of η and G_c : (a) Different values of η ; (b) Different values of G_c

4. Simulation of multiple crack interactions

This chapter illustrates the applicability of current phase-field fracture models in capturing the interactions among multiple cracks in problems involving pure displacement fields, coupled temperature-displacement fields, and coupled electrochemical concentration-displacement fields.

4.1. Interaction of parallel cracks

In fracture mechanics theory, two closely spaced parallel propagating cracks will interact, causing a deviation in their propagation directions. This section examines the ability of the phase-field method to capture crack interactions through this problem. As shown in Figure 8, a flat plate with length V and width H has a pre-existing crack symmetrically distributed along the left boundary, with a spacing of $2B$ and a length of a . When uniform tensile forces are applied to both ends of the plate, the cracks will continue to propagate. When the initial spacing between the two cracks differs, the subsequent initial crack propagation angles also differ. M.B. CIVELEK et al. [21] conducted a theoretical analysis of this problem and obtained a theoretical solution for the initial crack propagation angle.

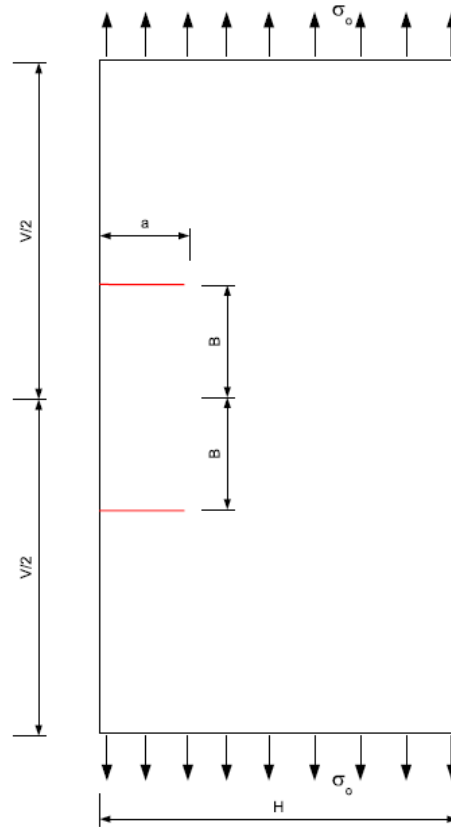


Figure 8. Geometry and boundary conditions for the problem of interacting parallel cracks

In this case study, due to the symmetry of the model, a symmetric modeling approach was first adopted to reduce the computational load. The material parameters were selected as $E = 208$ GPa, $\nu = 0.30$, and $G_c = 2.7$ N/mm². Figure 9 shows the initial crack propagation angles obtained for different initial crack spacings. The smaller the initial crack spacing, the larger the initial deflection angle. When $\frac{B}{H} = 0.2$ and 0.3 , the angles

between the crack propagation direction and the horizontal are 10° and 4° , respectively. When $\frac{B}{H} = 0.5$, the propagation of the two cracks no longer influences each other. The crack deflection angles obtained from the numerical calculations agree with the theoretical solutions of M.B. CIVELEK et al. [21], as shown in Figure 10.

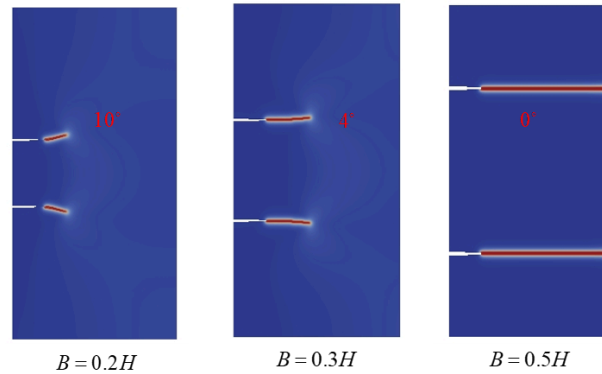


Figure 9. Crack propagation directions calculated using different geometric B -values

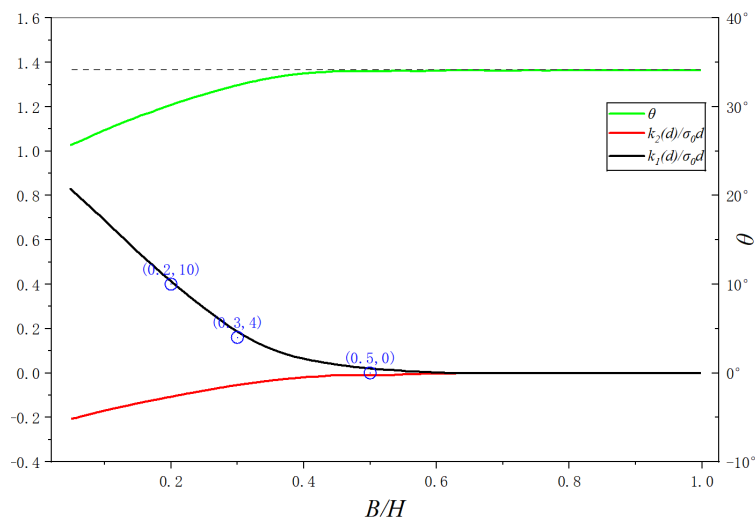


Figure 10. Comparison of the calculated crack propagation direction with the theoretical values reported by M.B. CIVELEK et al. [21]

At the same time, a simulation was performed on the entire specimen model using a non-perfectly symmetric mesh along the upper and lower surfaces of the specimen; that is, although the specimen as a whole was discretized using a regular quadrilateral mesh, the overall modeling could not guarantee exact symmetry between the upper and lower meshes. The calculation results for the $B = 0.2H$ specimen are shown in Figure 11. At the initial time, the discontinuous damage distribution between the two cracks is essentially identical with no significant differences; however, as loading continues, only one crack propagates, while the other is masked due to stress relief.

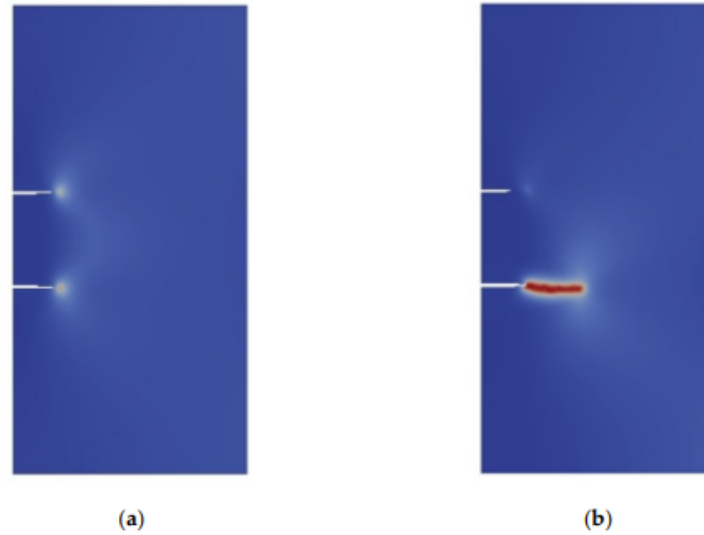


Figure 11. Phase-field distributions at different time steps for a non-perfectly symmetric mesh when $B = 0.2H$: (a) Initial time; (b) Crack propagation during specimen fracture

4.2. Glass plate quenching experiment

Quenching experiments are commonly used to investigate the thermal shock resistance of brittle materials. Geyer and Nasser [22] studied the interaction of multiple propagating cracks during the quenching of glass. As shown in Figure 12, a rectangular glass plate at room temperature (22 °C) contains initial cracks of 0.7 cm in depth spaced 2.5 cm apart along the edge of its long side. This edge is immersed in a coolant at -78 °C, and crack propagation occurs under the action of non-uniform thermal stresses. The cracks clearly exhibit two hierarchical levels: some cracks continue to propagate under thermal stress, while others are arrested. The spacing of the second-level cracks is approximately twice that of the first-level cracks.

For the same configuration, the quenching process of the rectangular glass plate is simulated. The material parameters are $E = 69$ GPa, $\nu = 0.23$, $G_c = 25.99$ J/m², $\alpha = 8.5 \times 10^{-6}$ K⁻¹. In the phase-field simulation, $\eta = 1 \times 10^{-6}$ N·s/mm², and plane stress conditions are assumed. As shown by Bourdin et al., inertial forces do not play a dominant role in crack propagation during this thermal shock process; therefore, a steady-state process is considered here. The stress field and phase field depend only on the temperature distribution, rather than on the specific time history of the heat conduction process. Accordingly, the time t is normalized by the thermal conduction parameters of the material as $\tau = \rho c_T L^2 / k_0$, where L is the height of the plate. To ensure sufficient nucleation of small cracks, l_0 is set to 1 mm, and the mesh size is set to 0.2 mm.

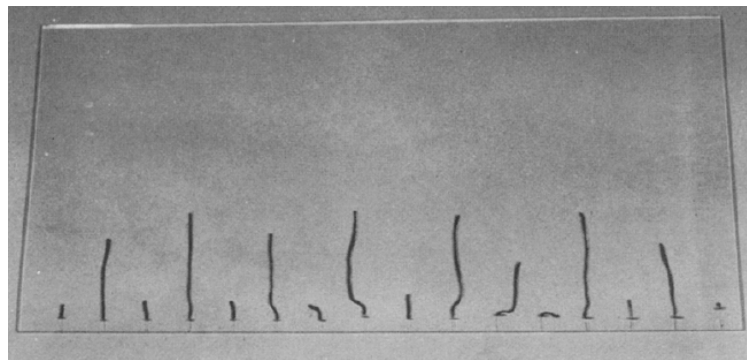


Figure 12. Results of crack propagation tests during the quenching of glass plates [22]

The computation results are shown in Figure 13, which illustrates the temperature and phase-field distributions within the glass plate at different time steps. It can be seen that, under conditions of rapid boundary cooling, crack nucleation occurs near the boundary due to non-uniform thermal stress. A total of 13 cracks formed, which is close to the number of initial cracks observed in the experiment. As the temperature inside the plate cooled, the cracks gradually propagated. However, due to the gradual decrease in the temperature gradient, local thermal stresses weakened, and the crack driving force diminished. Consequently, not all cracks continued to propagate simultaneously, and their propagation rates slowed. Instead, only about half of the cracks continued to propagate, with propagating cracks and stalled cracks alternating in distribution. The simulation results showed good agreement with the experimental results. The slight curvature of the cracks observed in the experiments is related to the non-ideal heterogeneity of the actual test material properties. In the simulation results, the crack propagation on both sides deviates toward the boundaries as a result of the cracks seeking maximum driving force and fracture energy release.

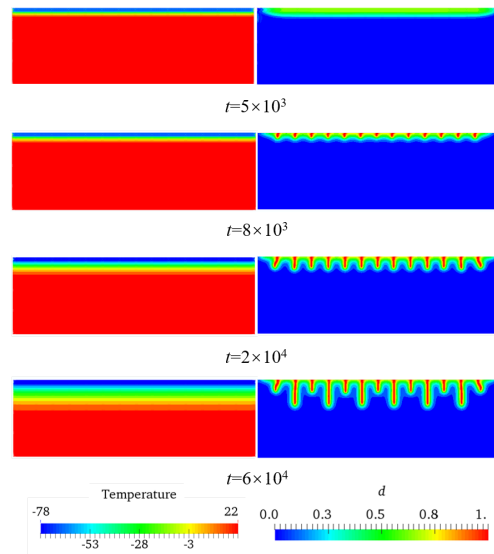


Figure 13. Temperature and phase-field distributions of the glass plate at different times

4.3. Crack propagation during the discharge process of lithium electrodes

Lithium electrodes possess a high capacity for lithium absorption, and they are highly susceptible to fracture during charge and discharge cycles due to the incompatible deformation caused by large variations in lithium concentration. During the charging of single-crystal lithium electrodes, lithium flux exhibits anisotropy. To delay fracture during the expansion associated with charging, a notch can be introduced at the boundary. This section presents a fracture analysis of a cylindrical electrode with a pre-existing notch during the discharge process. For simplicity, the problem is modeled as plane strain. The radius of the electrode cross-section is 10 mm. At the initial instant, the lithium concentration in the electrode is 1, and a zero-concentration boundary condition is applied at the outer boundary. The material parameters are $E = 80$ GPa, $\nu = 0.25$, $G_c = 1000$ J/m², $\theta = 0.06$. In the phase-field simulation, $\eta = 2 \times 10^{-3}$ N·s/mm². Since the discharge of the electrode is a slow process, inertial forces likewise do not play a dominant role in crack propagation; therefore, a steady-state process is considered here. The stress field and phase field depend only on the concentration distribution. Accordingly, the time t is normalized by the diffusion parameters of the material as $\tau = R^2/D_0$, where R is the radius of the electrode cross-section. To ensure sufficiently short crack initiation, 10 is set to 0.2 mm, and the mesh size is set to 0.07 mm.

The computed lithium concentration distributions and crack propagation patterns at different times are shown in Figure 14 and 15, respectively. From the results, it can be observed that as lithium ions are released, 12 cracks nucleate at the boundary and at the notch, with relatively uniform spacing. Notably, crack initiation occurs at the notch tip even before the concentration there changes significantly. This is caused by the stress concentration induced by the incompatible deformation of the outer material. As the lithium diffusion front advances, the cracks between the notches do not propagate further and are shielded. This is determined by the reduction in the crack driving force, which results from both the decrease in the local concentration gradient and the stress release caused by crack propagation at the notch. The cracks at the notch follow the lithium diffusion front closely from $t = 3$ to $t = 6$. After $t = 6$, the lithium ions in the central region of the electrode have diffused substantially, the concentration gradient decreases significantly, and crack propagation gradually stagnates. At $t = 18$, the lithium ions are fully released, no concentration gradient remains inside the electrode, the stress is relieved, and the brittle opening cracks undergo closure.

Through the above examples, the present multi-field coupled phase-field fracture model is shown to be capable of reasonably capturing the interaction and propagation of multiple cracks induced by heat conduction and electrochemical diffusion processes.

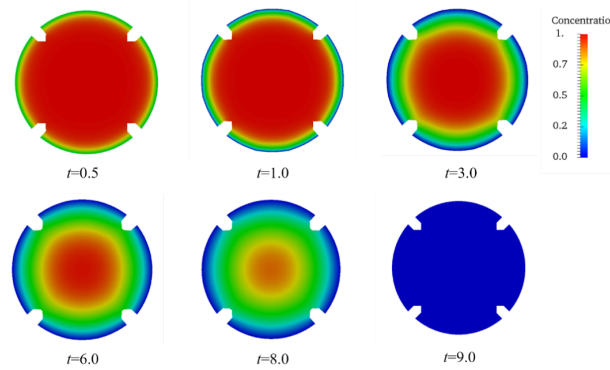


Figure 14. Lithium concentration evolution during the discharge of a lithium electrode

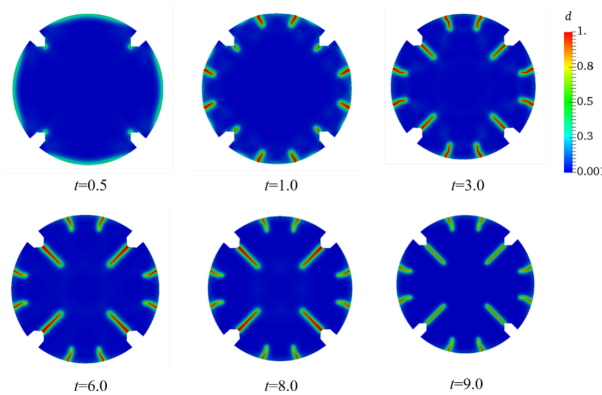


Figure 15. Crack propagation in the cross-section of a lithium electrode at different time points

5. Conclusions

This work establishes a simulation method for multi-field coupled multi-crack interaction based on the phase-field fracture approach. Cracks are represented in a diffuse manner through a phase-field variable, and the

influence of fracture damage on thermal conductivity and diffusivity is incorporated. Unified governing equations are formulated for thermo-mechanical coupling and chemo-mechanical coupling, and an implicit finite element method is employed for numerical solution. The model is validated through representative examples, and the interaction mechanisms of multiple cracks under multi-field coupling conditions are systematically investigated. The main conclusions are as follows:

(1) The accuracy of the phase-field fracture model is verified through mode I crack propagation and three-point bending fracture examples. Parametric analysis shows that the crack width control parameter affects the width of the damage zone at the crack tip and the ultimate load-carrying capacity. The viscosity coefficient influences the crack propagation speed and numerical stability; a smaller value approximates ideal brittle fracture. The critical energy release rate has a significant effect on the fracture limit and the crack propagation process.

(2) In the simulation of parallel crack interaction, the model accurately captures the crack deflection angle, which agrees well with the analytical solution. When the mesh is not fully symmetric, only one crack propagates while the other is shielded due to stress release, revealing the energy competition mechanism in multi-crack systems.

(3) In the simulation of glass plate quenching, the model successfully reproduces the nucleation, selective propagation, and arrest of multiple cracks under thermal shock. The crack number and propagation pattern are consistent with experimental observations. As the temperature gradient decreases, some cracks lose sufficient driving force and stop propagating; only about half of the cracks continue to grow, and they are distributed alternately with the arrested ones, reflecting the competition and shielding effects among cracks under thermo-mechanical coupling.

(4) In the simulation of the lithium electrode discharge process, the model reasonably describes crack nucleation and propagation driven by concentration-displacement coupling. Multiple cracks nucleate at the boundary and the notch, and the notch tip initiates cracking first due to stress concentration caused by incompatible deformation. As the diffusion front advances, the cracks between the notches are shielded, while the notch cracks follow the diffusion front closely and eventually stagnate or even close as the concentration gradient vanishes, revealing the distinctive behavior of multiple crack propagation under chemo-mechanical coupling.

In summary, the multi-field coupled phase-field fracture model developed in this work can effectively simulate the interaction of multiple cracks during heat conduction and electrochemical diffusion processes, providing a powerful numerical tool for the safety assessment of engineering structures in multi-field environments. Future work may further incorporate dynamic effects, plastic deformation, three-dimensional complex configurations, and coupling with additional physical fields such as electromagnetic fields, so as to extend the applicability of the model.

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